Primal-Dual Method

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Overview

Classical Primal-Dual Method

Primal LP problem:

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Dual problem:

$$\begin{aligned} \max \quad & b^T y \\ s.t. \quad & A^T y \leq c \\ & y \geq 0 \end{aligned}$$

Optimal solutions for primal and dual problems satisfy complementary slackness conditions(CSC):

$$y_i(A_i x - b_i) = 0$$

$$x_j(A^j y - c_j) = 0$$

A_i: the i th row of A
A^j: the j th column of A

Given a feasible dual solution y
for some $y_i > 0$, $A_i x - b_i = 0$; for $A^j y - c_j < 0$, we have $x_i = 0$.
$I = \{i | y_i = 0\}, J = \{j | A^j y - c_j = 0\}$
restricted primal problem:

$$
\begin{aligned}
z = \min \quad & \sum_{i \notin I} s_i + \sum_{j \notin J} x_j \\
s.t. \quad Ax_i \geq b_i & \qquad i \in I \\
Ax_i - s_i = b_i & \qquad i \notin I \\
x \geq 0 & \\
s \geq 0 &
\end{aligned}
$$

if $z = 0$, the primal feasible solution x obeys the CSC, x is the optimal solution; if $z \neq 0$, x violates some primal constrains or some CSC. y is less than OPT of dual.

dual of restricted primal problem:

$$\begin{aligned} \max \quad & b^T y' \\ s.t. \quad & A^j y' \leq 0 \qquad j \in J \\ & A^j y' \leq 1 \qquad j \notin J \\ & y'_i \geq -1 \qquad i \notin I \\ & y'_i \geq 0 \qquad i \in I \end{aligned}$$

since OPT of restricted primal > 0, there is a dual solution $y' \quad s.t. \quad b^T y' > 0$.
$y'' = y + \epsilon y'$, where $\epsilon \leq \min_{i \notin I : y'_i < 0} -y_i / y'_i$ and $\epsilon \leq \min_{j \notin J : A^j y' > 0} \frac{c_j - A^j y}{A^j y'}$
repeat this process until OPT of restricted primal problem is 0.

2 problems with classic primal-dual method:

- linear programming
- how to find a solution y' for dual of restricted primal problem.

Hitting set problem:
Hitting set is an equivalent reformulation of Set Cover.
Given subsets $T_1, \ldots, T_p$ of a ground set E and given a nonnegative cost c_e for every element in E, find a minimum-cost subset A s.t. $A \cap T_i \neq \emptyset$ for $i = 1, \ldots, p$.

Examples:

- undirected s-t shortest path: $\delta(S)$ needs to be hit if $s \in S, t \notin S$.
- minimum spanning tree: $\delta(S)$ needs to be hit for all S.

Hitting set problem can be formulated with integer programming(IP).

IP for undirected s-t shortest path:

$$\min \quad \sum_{e \in E} c_e x_e$$
$$s.t. \quad \sum_{e \in \delta(S)} x_e \geq f(S) \quad S \subset V$$
$$x_i \in \{0, 1\}$$

dual of its LP relaxation:

$$\max \quad \sum_{S: f(S)=1} y_S$$
$$s.t. \quad \sum_{S: e \in \delta(S)} y_S \leq c_e \quad e \in E$$
$$y_S \geq 0$$

* $\delta(S)$: a cut on S and $V - S$
* $A = \{e : x_e = 1\}$
* y: dual variable
* $T_1, \ldots, T_p$ sets to be hit

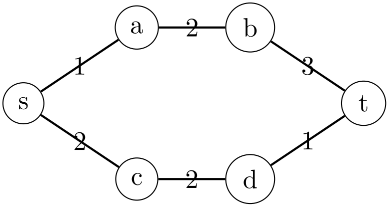
start with $y = 0$, if there is any $\delta(S) : f(S) = 1$ that $|A \cap \delta(S)| = 0$, we increase the corresponding dual variable y_S:

$$y_S = min_{e \in \delta(S)} \{c_e - \sum_{T \neq S : e \in \delta(T)} y_T\}$$

if $\sum_{S:e \in \delta(S)} y_S \leq c_e$ is satisfied, set corresponding primal variable $x_e = 1$.
some edge $e \in \delta(S)$ will be add to A.

$y \leftarrow 0$
$A \leftarrow \emptyset$
While $\exists k : A \cap T_k = \emptyset$
  Increase y_k until $\exists e \in T_k : \sum_{i:e \in T_i} y_i = c_e$
  $A \leftarrow A \cup \{e\}$
Output A (and y)

Example



cuts	1	2	3	4	5
sa,sc	+1(sa)				
sa,cd					
sa,dt					
ab,sc		+1(sc)			
ab,cd			+1(ab)		
ab,dt					
bt,sc					
bt,cd				+1(cd)	
bt,dt					+1(dt)

design rules

- reverse delete step
- minimal violated set rule
- uniorm increase rule

$y \leftarrow 0$
$A \leftarrow \emptyset$
$l \leftarrow 0$
While A is not feasible
    $l \leftarrow l + 1$
    $\mathcal{V} \leftarrow$ VIOLATION(A)
    Increase y_k uniformly for all $T_k \in \mathcal{V}$ until $\exists e_l \notin A : \sum_{i: e_l \in T_i} y_i = c_{e_l}$
    $A \leftarrow A \cup \{e_l\}$
For $j \leftarrow l$ downto 1
    if $A - \{e_j\}$ is feasible then $A \leftarrow A - \{e_j\}$
Output A (and y)

approximation rate

$$
\begin{aligned}
c(A) &= \sum_{e \in A} c_e \\
&= \sum_{e \in A} \sum_{i: e \in T_i} y_i \\
&= \sum_{i=1}^{p} |A \cap T_i| y_i
\end{aligned}
$$

let $\alpha = \max\{|A \cap T_i|\}$. Since $\sum y_i \leq OPT$, we get

$$
c(A) \leq \sum_{i=1}^{p} \alpha y_i \leq \alpha OPT
$$

Define **minimal augmentation** B of an infeasible solution A: B is a feasible solution that includes A and for any subset $C \subset B - A$, $A \cup C$ is not a feasible solution.

For any final primal solution A_f, $|B \cap T_i| \geq |A_f \cap T_i|$ holds if B is the maximum minimal augmentation set.

$$\beta = \max_{A:\exists T_i:|T_i \cap A|=0} \max_{B} |B \cap T(A)|$$

$T(A)$ is the T_i selected by the algorithm for infeasible solution A.

Consider the violation set $\mathcal{V}_j$, $y_i = \sum_{j:\, T_i \in \mathcal{V}_j} \epsilon_j$
suppose now there is p violated cuts and l violated sets.

$$\sum_{i=1}^{p} y_i = \sum_{i=1}^{p} \sum_{j:\, T_i \in \mathcal{V}_j} \epsilon_j$$
$$= \sum_{j=1}^{l} |\mathcal{V}_j| \epsilon_j$$

$$\sum_{i=1}^{p} |A_f \cap T_i| y_i = \sum_{i=1}^{p} |A_f \cap T_i| \sum_{j:\, T_i \in \mathcal{V}_j} \epsilon_j$$
$$= \sum_{j=1}^{l} \Big(\sum_{T_i \in \mathcal{V}_j} |A_f \cap T_i| \Big) \epsilon_j$$

Compare $\sum_{j=1}^{l}(\sum_{T_i \in \mathcal{V}_j} |A_f \cap T_i|)\epsilon_j$ with $\sum_{j=1}^{l} |\mathcal{V}_j|\epsilon_j$.

$$\sum_{i=1}^{p} |A_f \cap T_i| \leq \gamma |\mathcal{V}_j|$$

then

$$\begin{aligned}
\sum_{i=1}^{p} |A_f \cap T_i| y_i &= \sum_{j=1}^{l} (\sum_{T_i \in \mathcal{V}_j} |A_f \cap T_i|)\epsilon_j \\
&\leq \sum_{j=1}^{l} \gamma |\mathcal{V}_j| \epsilon_j \\
&= \gamma \sum_{i=1}^{p} y_i
\end{aligned}$$

Again consider minimal augmentation B:
$\max |B \cap T_i| \geq |A_f \cap T_i|$ holds, change $|A_f \cap T_i|$ to $|B \cap T_i|$:

$$\sum_{i=1}^{p} |A_f \cap T_i| \leq \sum_{T_i \in \mathcal{V}(A)} |B \cap T_i| \leq \gamma |\mathcal{V}(A)|$$

γ is the approximation rate.

Example

prove that the algorithm for s-t shortest path problem above gives the optimal solution.

$$\beta = \max_{A:\exists T_i:|T_i \cap A|=0} \max_B |B \cap T(A)|$$

the algorithm considers only one infeasible cut $\delta(S)$ and $s \in S$ and S is minimal. After increasing the corresponding dual variable only one edge will be added to A, so the minimal augmentation $B \cap T(A) = 1$.

0-1 Function f(S)

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ s.t. \quad & \sum_{e \in \delta(S)} x_e \geq f(S) \quad S \subset V \\ & x_i \in \{0,1\} \end{aligned}$$

To formulate another problem, only need to change the definition of $f(S)$.

0-1 function: $f : 2^V \rightarrow \{0,1\}$

properties

Maximality: If $A \cap B = \emptyset$, $f(A \cup B) \leq \max(f(A), f(B))$.
A violated set(cut) for A is a connected component of $G(V, A)$

downward monotone: If $f(S) \leq f(T)$ for all $S \supseteq T \neq \emptyset$.

0-1 proper function:

- $f(V) = 0$
- f satisfies the Maximality property.
- $f(S) = f(V - S)$ for all $S \subseteq V$

downward monotone function

Example: edge-covering problem:
$f(S) = 1$ iff. $|S| = 1$. satisfies downward monotone property.

Primal-Dual method gives a 2-approximation algorithm for edge covering. (use $\beta = \max_{A:\exists T_i:|T_i \cap A|=0} \max_B |B \cap T(A)|$)

Theorem 1 Primal-Dual algorithm gives a 2-approximation algorithm with any downward monotone function.

$$\sum_{S \in \mathcal{V}(A)} |B \cap \delta(S)| \leq \gamma |\mathcal{V}(A)|$$

prove $\gamma = 2$

construct a graph H by taking the graph (V, B) and shrinking the connected component of (V, A).(H is a forest)
$W = \{w | f(S_w) = 1\}$

$$\sum_{v \in W} d_v \leq 2|W|$$

Lemma connected components in H has at most one vertex v such that $f(S_v) = 0$.
(prove by contradiction)

c is the number of connected components in H.

$$\sum_{v \in W} d_v \leq \sum_{v \in H} d_v = 2(|H| - c) \leq 2|W|$$

0-1 proper function

complementarity for 0-1 proper function which satisfies Maximality property, if $f(S) = f(A) = 0$ for $A \in S$ then $f(S - A) = 0$.

proof Suppose $f(S - A) = 1$.
$f(V - S) = f(S) = 0, f(V - A) = f(A) = 0, f(V - S + A) = f(S - A) = 1$. Function f satisfies Maximality property: $1 = f(V - S + A) \leq \max(f(V - A), f(A)) = 0$, a contradiction.

0-1 proper function

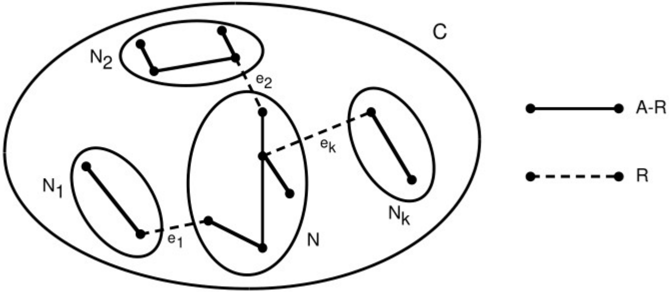
Example generalized steiner tree: minimum forest that connects all vertices T_i for
$i = 1, \ldots, p$.
$f(S) = 1$ if $\exists i \in \{1, 2, \ldots, p\}$ s.t. $S \cup T_i \neq \emptyset$ and $S \cup T_i \neq T_i$.

$\beta = \max_{A: \exists T_i : |T_i \cap A| = 0} \max_B |B \cap T(A)|$. β can be $|V| - 1$.(Consider a complete graph. Each edge has cost 1. At the first step of primal dual algorithm, $A = \emptyset$,
$|B \cap T(A)| = |V| - 1$)

Theorem 2 Primal-dual algorithm gives a 2-approximation algorithm for IP with any 0-1 proper function.

Lemma 1 f is 0-1 proper function, A is any feasible solution. $R = \{e | A - e \text{ is feasible}\}$, then $A - R$ is feasible.

proof



Lemma 2 No leaf v of H satisfies $f(S_v) = 0$.

proof Suppose some leaf v satisfies $f(S_v) = 0$, let C be the connected component of (V, B) that contains S_v. Since B is feasible, $f(C) = 0$. Since $f(S_v) = 0$ and by complementarity property, $f(C - S_v) = 0$. But there is an edge in B that connects $C - S_v$ and S_v, B is the **minimal augmentation**, a contradiction.

Proof of Theorem 2.

By Lemma 2, all vertices with degree 1 are in W.

$$\sum_{v \in W} d_v = \sum_{v \in H} d_v - \sum_{v \notin W} d_v \leq 2(|H| - 1) - 2(|H| - |W|) = 2|W| - 2$$

Thank you